Rate of convergence of geometric symmetrizations

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Abstract

It is a classical fact, that given an arbitrary convex body $K \subset \mathbb{R}^n$, there exists an appropriate sequence of Minkowski symmetrizations (or Steiner symmetrizations), that converges in Hausdorff metric to a Euclidean ball. Here we provide quantitative estimates regarding this convergence, for both Minkowski and Steiner symmetrizations. Our estimates are polynomial in the dimension and in the logarithm of the desired distance to a Euclidean ball, improving previously known exponential estimates. Inspired by a method of Diaconis [D], our technique involves spherical harmonics. We also make use of an earlier result by the author regarding "isomorphic Minkowski symmetrization".

1 Introduction

Let $K \subset \mathbb{R}^n$ be a convex body, and denote by $|\cdot|$ and $\langle\cdot,\cdot\rangle$ the usual Euclidean norm and scalar product in $\mathbb{R}^n$. Given a vector $u \in S^{n-1} = \{x \in \mathbb{R}^n; |x| = 1\}$, we denote by $\pi_u(x) = x - 2\langle x, u \rangle u$ the reflection operator with respect to the hyperplane through the origin, which is orthogonal to u in $\mathbb{R}^n$. The result of a Minkowski symmetrization (sometimes called Blaschke symmetrization) of K with respect to u, is the body

$$\tau_u(K) = \frac{K + \pi_u(K)}{2}$$

where the Minkowski sum of two sets $A, B \subset \mathbb{R}^n$ is defined as $A + B = \{a + b; a \in A, b \in B\}$. Let h_K denote the supporting functional of K, i.e. for $u \in \mathbb{R}^n$

$$h_K(u) = \sup_{x \in K} \langle x, u \rangle.$$

Then $h_{\tau_u(K)}(v) = \frac{1}{2}[h_K(v) + h_K(\pi_u(v))]$. The mean width of K is defined as $w(K) = 2M^*(K) = 2\int_{S^{n-1}} h_K(u)d\sigma(u)$, where σ is the unique rotation invariant probability measure on the sphere. The mean width is preserved under Minkowski symmetrizations.

Steiner symmetrization of K with respect to a hyperplane H yields the unique body $S_H(K)$ such that for any line l perpendicular to H,

(i) $S_H(K) \cap l$ is a closed segment whose center lies on H.

(ii) $Meas(K \cap l) = Meas(S_H(K) \cap l)$.

where $Meas$ is the one dimensional Lebesgue measure in the line l. Steiner symmetrization preserves the volume of a set and transforms convex sets to convex sets. See e.g. [BF] for more information about these symmetrizations, and their applications in proving geometric inequalities.

Consecutive Minkowski/Steiner symmetrizations may cause a convex body to resemble a Euclidean ball. Starting with an arbitrary convex body, one may apply a suitable sequence of Minkowski/Steiner symmetrizations, and obtain a sequence of bodies that converges to a Euclidean ball. This Euclidean ball would have the same mean width/volume as had the original body. In this note, we investigate the rate of this convergence. We ask how many symmetrizations are needed, in order to transform an arbitrary convex body $K \subset \mathbb{R}^n$ into a body that is ε-close to a Euclidean ball. Our question is "almost isometric" in its nature, as we try to provide reasonable estimates even for small values of ε. Previous results in the literature are mostly of "isomorphic" nature, in the sense that the symmetrization process is aimed at obtaining a body which is uniformly "isomorphic" to a Euclidean ball (a body is "isomorphic" to a Euclidean ball if its distance to a Euclidean ball is bounded by some fixed, universal constant).

The first quantitative result regarding Minkowski symmetrization appears in [BLM1]. Denote by D the standard Euclidean ball in $\mathbb{R}^n$. Their result reads as follows:

Theorem 1.1 *Let $0 < \varepsilon < 1$, $n > n_0(\varepsilon)$. Given an arbitrary convex body $K \subset \mathbb{R}^n$, there exist $cn\log n + c(\varepsilon)n$ Minkowski symmetrizations that transform K into a body $\tilde{K}$ such that*

$$(1-\varepsilon)M^*(K)D \subset \tilde{K} \subset (1+\varepsilon)M^*(K)D$$

where $c(\varepsilon), n_0(\varepsilon)$ are of the order of $exp(c\varepsilon^{-2}|\log\varepsilon|)$ and $c > 0$ is a numerical constant.

Their proof uses the method of random Minkowski symmetrizations. In [K2], the notion of randomness was altered, and has lead to an improvement of the dependence on the dimension n. The following is proved in [K2]:

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Theorem 1.2 *Let $n \geq 2$ and let $K \subset \mathbb{R}^n$ be a convex body. Then there exist $5n$ Minkowski symmetrizations, such that when applied to K, the resulting body $\tilde{K}$ satisfies,*

$$\left(1 - c\frac{|\log\log n|}{\sqrt{\log n}}\right) M^*(K)D \subset \tilde{K} \subset \left(1 + c\frac{|\log\log n|}{\sqrt{\log n}}\right) M^*(K)D$$

where $c > 0$ is some numerical constant.

Note that both in [K2] and in [BLM1], for any fixed dimension, one cannot even formally conclude that there is convergence to a Euclidean ball. This note fills that gap in the literature, and also provides surprisingly good dependence on ε. The following theorem is proved here:

Theorem 1.3 *Let $n \geq 2$, $0 < \varepsilon < \frac{1}{2}$, and let $K \subset \mathbb{R}^n$ be a convex body. Then there exist $cn\log\frac{1}{\varepsilon}$ Minkowski symmetrizations, that transform K into a body $\tilde{K}$ that satisfies*

$$(1 - \varepsilon)M^*(K)D \subset \tilde{K} \subset (1 + \varepsilon)M^*(K)D$$

where $c > 0$ is some numerical constant.

Our approach to the problem of Minkowski symmetrization involves a number of novel ideas. First, rather than applying random Minkowski symmetrizations, at each step we apply n symmetrizations with respect to the vectors of some random orthonormal basis. This change of randomness improves the rate of convergence by a factor of $\log n$ (see [K1], [K2] and also the remark following Corollary 3.3 here). Second, the use of spherical harmonics allows us to obtain good estimates regarding symmetrization of polynomials on the sphere. Finally, we approximate the supporting functional of K with an appropriate polynomial (applying Theorem 1.2 and a Jackson type theorem), and use the estimates obtained for symmetrization of polynomials.

Quantitative estimates regarding Steiner symmetrization are more difficult to obtain, as the problem is non-linear. The earliest estimate in the literature is due to Hadwiger [H]. It gives an estimate of the order of $\left(c\frac{\sqrt{n}}{\varepsilon^2}\right)^n$ for the number of Steiner symmetrizations required in order to transform an arbitrary n-dimensional convex body, to become ε-close to a Euclidean ball. In addition, an isomorphic result appears in [BLM2], which was improved by a logarithmic factor in [KM]. The following is proved in [KM]:

Theorem 1.4 *Let $n \geq 2$ and let $K \subset \mathbb{R}^n$ be a convex body, with $Vol(K) = Vol(D)$. Then there exist $3n$ Steiner symmetrizations, such*

that when applied to K, the resulting body $\tilde{K}$ satisfies,

$$cD \subset \tilde{K} \subset CD$$

where $c, C > 0$ are some numerical constants.

Some related estimates also appear in [T]. Our result is the first estimate which is polynomial in n and in $\log \frac{1}{\varepsilon}$. This shows that the precise geometric shape of a convex body cannot prevent fast symmetrization of the body into an almost Euclidean ball. In this note we shall prove the following theorem.

Theorem 1.5 *Let $K \subset \mathbb{R}^n$ be a convex body, and let $0 < \varepsilon < \frac{1}{2}$. Let $r > 0$ be such that $Vol(K) = Vol(rD)$. Then there exist $cn^4 \log^2 \frac{1}{\varepsilon}$ Steiner symmetrizations, that transform K into a body $\tilde{K}$ that satisfies*

$$(1 - \varepsilon)rD \subset \tilde{K} \subset (1 + \varepsilon)rD$$

where $c > 0$ is some numerical constant.

The powers of n and $\log \frac{1}{\varepsilon}$ in Theorem 1.5 seem non optimal. We conjecture that $cn \log \frac{1}{\varepsilon}$ Steiner symmetrizations are sufficient. Regarding Minkowski symmetrizations, our result is tight in the sense that the powers in Theorem 1.3 cannot be improved.

The proof of Theorem 1.5 is an application of Theorem 1.3 and of a geometric result by Bokowski and Heil. Throughout this paper, we denote by c, C, c' etc. positive numerical constants whose value is not necessarily equal in different appearances.

2   Spherical Harmonics

In this section we summarize a few facts about spherical harmonics, to be used later on. For a comprehensive discussion on the subject, we refer the reader to the concise expositions in [SW], chapter $IV.2$, in [M] and in [G]. $P_k : \mathbb{R}^n \to \mathbb{R}$ is a homegeneous harmonic of degree k, if P_k is a homogeneous polynomial of degree k in $\mathbb{R}^n$, and P_k is harmonic (i.e. $\triangle P_k \equiv 0$). We denote,

$\mathcal{S}_k = \{P|_{S^{n-1}} \; ; \; P : \mathbb{R}^n \to \mathbb{R} \text{ is a homogenous harmonic of degree } k\}$

where $P|_{S^{n-1}}$ is the restriction of the polynomial P to the sphere. $\mathcal{S}_k$ is the space of spherical harmonics of degree k. It is a linear space of dimension $\frac{(2k+n-2) \; (n+k-3)!}{k! \; (n-2)!}$. For $k \neq k'$, the spaces $\mathcal{S}_k$ and $\mathcal{S}_{k'}$ are orthogonal to each other in $L_2(S^{n-1})$. In addition, if P is a polynomial of degree k in $\mathbb{R}^n$, then $P|_{S^{n-1}}$ can be expressed as a sum of spherical harmonics of degrees not larger than k. Therefore, $L_2(S^{n-1}) = \bigoplus_k \mathcal{S}_k$.

Spherical harmonics possess many symmetry properties, partly due to their connection with the representations of $O(n)$ (e.g. [V], chapter 9). For a fixed dimension n, the Gegenbauer polynomials $\{G_i(t)\}_{i=0}^{\infty}$ are defined by the following three conditions:

(i) $G_i(t)$ is a polynomial of degree i in one variable.

(ii) For any $i \neq j$ we have $\int_{-1}^{1} G_i(t)G_j(t)\left(1-t^2\right)^{\frac{n-3}{2}} dt = 0$.

(iii) $G_i(1) = 1$ for any i.

The Gegenbauer polynomials are closely related to spherical harmonics. Next, we reformulate Lemma 3.5.4 from [G], which is credited to Schneider. This useful lemma also follows from Corollary 2.13, chapter IV of [SW], and is true for all $n \geq 2$.

Lemma 2.1 *Let $g \in \mathcal{S}_k$ be such that $\|g\|_2^2 = \int_{S^{n-1}} g^2(x)d\sigma(x) = 1$. Then,*

$$\int_{O(n)} g(U^{-1}x)g(U^{-1}y)d\mu(U) = G_k(\langle x,y \rangle)$$

where μ is the Haar probability measure on $O(n)$.

The following lemma reflects the fact that $\mathcal{S}_k$ is an irreducible representation space of $O(n)$. We denote by $Proj_{\mathcal{S}_k} : L_2(S^{n-1}) \to \mathcal{S}_k$ the orthogonal projection onto $\mathcal{S}_k$.

Lemma 2.2 *Let $f \in L_2(S^{n-1})$, and let $g \in \mathcal{S}_k$ be such that $\|g\|_2 = 1$. Then,*

$$\int_{O(n)} \left(\int_{S^{n-1}} f(Ux)g(x)d\sigma(x) \right)^2 d\mu(U) = \frac{\|Proj_{\mathcal{S}_k}(f)\|_2^2}{dim(\mathcal{S}_k)} \qquad (1)$$

where μ is the Haar probability measure on $O(n)$.

Proof: Let $\{g_1, .., g_N\}$ be an orthonormal basis of $\mathcal{S}_k$. Then,

$$\sum_{i=1}^{dim(\mathcal{S}_k)} \int_{O(n)} \left(\int_{S^{n-1}} f(Ux)g_i(x)d\sigma(x) \right)^2 d\mu(U) \qquad (2)$$

$$= \int_{O(n)} \|Proj_{\mathcal{S}_k}(f \circ U)\|_2^2 d\mu(U) = \|Proj_{\mathcal{S}_k}(f)\|_2^2$$

because of the rotation invariance of $\mathcal{S}_k$. Therefore, it is sufficient to prove that the integral in (1) does not depend on the choice of $g \in \mathcal{S}_k$, as long as it satisfies $\|g\|_2 = 1$. Indeed, in that case each of the summands in (2) equals $\frac{\|Proj_{\mathcal{S}_k}(f)\|_2^2}{dim(\mathcal{S}_k)}$, for an arbitrary orthonormal basis $\{g_1, .., g_N\}$ of $\mathcal{S}_k$. Let us try to simplify the integral in (1):

$$\int_{O(n)} \int_{S^{n-1}} f(Ux)g(x)d\sigma(x) \int_{S^{n-1}} f(Uy)g(y)d\sigma(y)d\mu(U)$$

$$= \int_{S^{n-1}} \int_{S^{n-1}} f(x)f(y) \int_{O(n)} g(U^{-1}x)g(U^{-1}y)d\mu(U)d\sigma(x)d\sigma(y).$$

By Lemma 2.1, $\int_{O(n)} g(U^{-1}x)g(U^{-1}y)d\mu(U) = G_k(\langle x, y \rangle)$. Hence, the integral in (1) equals

$$\int_{S^{n-1}} \int_{S^{n-1}} f(x)f(y)G_k(\langle x, y \rangle)d\sigma(x)d\sigma(y)$$

which does not depend on g, and the lemma is proved. $\qquad\square$

3 Spherical Harmonics and Minkowski Symmetrization

In this section we apply a series of Minkowski symmetrizations to a convex body $K \subset \mathbb{R}^n$. Each step in the symmetrization process consists of symmetrizing K with respect to the n vectors of an orthonormal basis $\{e_1, .., e_n\}$ in $\mathbb{R}^n$. Such a step is denoted here as an "orthogonal symmetrization" with respect to $\{e_1, .., e_n\}$. Applying an "orthogonal symmetrization" with respect to $\{e_1, .., e_n\}$ to K, yields a body denoted by K'. Let h be the supporting functional of K, and h' be the supporting functional of K'. Then,

$$h'(x) = \mathbb{E}_\varepsilon \; h \left(\sum_{i=1}^{n} \varepsilon_i \langle x, e_i \rangle e_i \right) \tag{3}$$

where the expectation is over $\varepsilon \in \{\pm 1\}^n$, with respect to the uniform probability measure on the discrete cube. Note that by (3), orthogonal symmetrization may be viewed as an operation on support functions, rather than on convex bodies. Furthermore, we may apply an "orthogonal symmetrization" to any function on the sphere, which is not necessarily a support function of a convex body. Next, we analyze the effect of orthogonal symmetrizations on spherical harmonics.

Let k be a positive integer. A function $g \in L_2(S^{n-1})$ is called "invariant with respect to the orthonormal basis $\{e_1, .., e_n\}$", if for any $\varepsilon \in \{\pm 1\}^n$, we have $g(x) = g\left(\sum_i \varepsilon_i \langle x, e_i \rangle e_i \right)$. For a fixed orthonormal basis $\{e_1, .., e_n\}$ in $\mathbb{R}^n$, we denote by $\mathcal{S}_k^0$ the linear space of all invariant functions in $\mathcal{S}_k$. Let $Proj_{\mathcal{S}_k^0} : \mathcal{S}_k \to \mathcal{S}_k^0$ be the orthogonal projection in $L_2(S^{n-1})$. Then for $g \in \mathcal{S}_k$,

$$g'(x) = \mathbb{E}_\varepsilon \; g \left(\sum_{i=1}^{n} \varepsilon_i \langle x, e_i \rangle e_i \right) \quad \Longleftrightarrow \quad g' = Proj_{\mathcal{S}_k^0}(g),$$

i.e. the orthogonal symmetrization of g is the projection of g onto $\mathcal{S}_k^0$.

Lemma 3.1 *If k is odd, $dim(\mathcal{S}_k^0) = 0$. Otherwise,*

$$dim(\mathcal{S}_k^0) = \binom{n + \frac{k}{2} - 2}{n - 2}.$$

Proof: The odd case is easy, since for $g \in \mathcal{S}_k$ we necessarily have $g(x) = -g(-x)$, and for $g \in \mathcal{S}_K^0$ we have $g(x) = g(-x)$. Hence, only $0 \in \mathcal{S}_k^0$. Next, assume that k is even, and let $g \in \mathcal{S}_k^0$ be an invariant polynomial with respect to the basis $\{e_1, .., e_n\}$. We use the coordinates $x_1, .., x_n$ with respect to this basis. Fixing $x_2, .., x_n$ the polynomial g satisfies $g_{x_2,..,x_n}(x_1) = g_{x_2,..,x_n}(-x_1)$, and hence only even degrees of x_1 occur in $g_{x_2,..,x_n}$. By repeating the argument for the rest of the variables, we get that g is a function of $x_1^2, .., x_n^2$ alone. We can write,

$$g(x_1, .., x_n) = \sum_{j=0}^{k/2} x_n^{2j} A_j(x_1, .., x_{n-1}) \tag{4}$$

where A_j is a homogeneous polynomial of degree $k - 2j$, which depends solely on $x_1^2, .., x_{n-1}^2$. Let us calculate the Laplacian of (4):

$$0 = \sum_{j=1}^{k/2} 2j(2j-1)x_n^{2j-2} A_j(x_1, .., x_{n-1}) + \sum_{j=0}^{k/2-1} x_n^{2j} \triangle A_j(x_1, .., x_{n-1})$$

or equivalently, $g \in \mathcal{S}_k^0$ if and anly if for all $0 \le j \le \frac{k}{2} - 1$,

$$(2j+2)(2j+1)A_{j+1} = -\triangle A_j. \tag{5}$$

Therefore we are free to choose A_0 any way we like, as long as it is a homogeneous polynomial of degree k, which involves only even powers of the $n - 1$ variables. When A_0 is fixed, A_1, A_2 etc. are determined by equation (5), and the function g is recovered.

Hence, $dim(\mathcal{S}_k^0)$ equals the dimension of the space of the possible $A_0(x_1^2, .., x_{n-1}^2)$, which is the dimension of the space of all homogeneous polynomials of degree $k/2$ in $n - 1$ variables. This number is known to be $\binom{n + \frac{k}{2} - 2}{n - 2}$. $\qquad\square$

We denote $N_k = dim(\mathcal{S}_k) = \binom{n + k - 2}{n - 2} \frac{n+2k-2}{n+k-2}$, and for an even k denote $N_k^0 = dim(\mathcal{S}_k^0) = \binom{n + k/2 - 2}{n - 2}$. Clearly, these two quantities depend on n which is absent from the notation, yet the appropriate value of n will be obvious from the context. We are now ready to calculate the L_2 norm of a "random orthogonal symmetrization" of a spherical harmonic - an orthogonal symmetrization with respect to

a basis that is chosen uniformly over $O(n)$. Clearly, any "orthogonal symmetrization" of an odd degree spherical harmonic vanishes. The even case is treated in the following proposition.

Proposition 3.2 *Let k be a positive even integer, and let $g \in \mathcal{S}_k$ be a spherical harmonic. We randomly select an orthonormal basis $\{v_1, .., v_n\} \in O(n)$, and symmetrize g with respect to this basis. Then,*

$$\mathbb{E}\|g'_{v_1,..,v_n}\|_2^2 = \frac{N_k^0}{N_k}\|g\|_2^2 < \left(\frac{k}{n-2+k}\right)^{k/2}\|g\|_2^2$$

where the expectation is over the random choice of $\{v_1, .., v_n\} \in O(n)$ (with respect to the Haar probability measure on $O(n)$).

Proof: Fix an orthonormal basis $\{e_1, .., e_n\}$ of $\mathbb{R}^n$, and consider $\mathcal{S}_k^0$ with respect to that basis. Fix also an orthonormal basis $S_1, .., S_{N_k^0}$ of $\mathcal{S}_k^0$. From the discussion before Lemma 3.1,

$$g'_{e_1,..,e_n} = Proj_{\mathcal{S}_k^0}(g)$$

and if the columns of $U \in O(n)$ are $\{v_1, .., v_n\}$, then

$$g'_{v_1,..,v_n} = \left(Proj_{\mathcal{S}_k^0}(g \circ U)\right) \circ U^{-1}.$$

Hence,

$$\|g'_{v_1,..,v_n}\|_2^2 = \|Proj_{\mathcal{S}_k^0}(g \circ U)\|_2^2 = \sum_{j=1}^{N_k^0}\left(\int_{S^{n-1}} g(Ux)S_j(x)d\sigma(x)\right)^2$$

and by Lemma 2.2,

$$\mathbb{E}\|g'_{v_1,..,v_n}\|_2^2 = \frac{\sum_{j=1}^{N_k^0}\|g\|_2^2}{N_k} = \frac{N_k^0}{N_k}\|g\|_2^2.$$

Note that

$$\frac{N_k^0}{N_k} = \frac{n+k-2}{n+2k-2}\prod_{i=1}^{k/2}\frac{(n+i-2)(k/2+i)}{(n+2i-3)(n+2i-2)} < \prod_{i=1}^{k/2}\frac{k/2+i}{k/2+i+n-2}$$

which lies between $\left(\frac{k/2}{n-2+k/2}\right)^{k/2}$ and $\left(\frac{k}{n-2+k}\right)^{k/2}$. $\qquad\square$

Since $\left(\frac{k}{n-2+k}\right)^{k/2}$ is a decreasing function of k, then $\left(\frac{k}{n-2+k}\right)^{k/2} \leq \frac{2}{n}$ for any $k \geq 2$, and we obtain the following corollary:

Corollary 3.3 *Let $f \in L_2(S^{n-1})$ satisfy $\int_{S^{n-1}} f(x)d\sigma(x) = 0$. We randomly select $\{v_1, .., v_n\} \in O(n)$. Then,*

$$\mathbb{E}\|f'_{v_1,..,v_n}\|_2 < \frac{c}{\sqrt{n}}\|f\|_2$$

where the expectation is taken over the choice of $\{v_1, .., v_n\} \in O(n)$, and $c = \sqrt{2}$.

Proof: Expand f into spherical harmonics: $f = \sum_{k=1}^{\infty} f_k$ where $f_k = Proj_{\mathcal{S}_k}(f)$. Then $f'_{v_1,..,v_n} = \sum_{k=1}^{\infty} (f_k)'_{v_1,..,v_n}$ and

$$\mathbb{E}\|f'_{v_1,..,v_n}\|_2^2 = \sum_{k=2}^{\infty} \mathbb{E}\| (f_k)'_{v_1,..,v_n} \|_2^2 \leq \sum_k \frac{2}{n}\|f_k\|_2^2 \leq \frac{2}{n}\|f\|_2^2.$$

An application of Jensen inequality concludes the proof. $\qquad\square$

Remark: Using similar methods, one can prove that if $g \in \mathcal{S}_k$ and $\tau_u(g)(x) = \frac{g(x)+g(\pi_u(x))}{2}$, then

$$\mathbb{E}_u\|\tau_u(g)\|_2^2 = \frac{n-2+k}{n-2+2k}\|g\|_2^2.$$

Note the advantage of symmetrizing with respect to the n vectors of a random orthonormal basis, compared to symmetrization with respect to n random sphere vectors. For instance, if $k = 2$ then

$$\left(\frac{n-2+k}{n-2+2k}\right)^n \approx \frac{1}{e^2}.$$

Hence n random symmetrizations may reduce the expectation of the L_2 norm only by a constant factor.

4   Decay of L_∞ norm

In Proposition 3.2 and Corollary 3.3 we established a sharp estimate for the decay of the L_2 norm under an "orthogonal symmetrization". Now we deal with the more difficult problem of estimating the decay of the L_∞ norm of the function. Our main tool is the following known lemma (see e.g. page 14 of [M]):

Lemma 4.1 *Let $g \in \mathcal{S}_k$ be a spherical harmonic of degree k. Then,*

$$\|g\|_\infty \leq \sqrt{dim(\mathcal{S}_k)}\|g\|_2 = \sqrt{N_k}\|g\|_2$$

where $\|g\|_\infty = \sup_{x \in S^{n-1}} |g(x)|$.

We make use of the following well-known estimate of binomial coefficients. For any $1 \le k \le n$,

$$\left(\frac{n}{k}\right)^k \le \binom{n}{k} < \left(e\frac{n}{k}\right)^k. \tag{6}$$

In the following combinatorial lemmas, "log" is to be understood as the natural logarithm.

Lemma 4.2 *Let $\varepsilon > 0$, $n \ge 3$, and let $k \ge 2$ be an integer. Then,*

$$N_k^{c_1 \frac{1+\log\left(1+\frac{2}{\varepsilon}\right)}{1+\log\left(1+\frac{k}{n}\right)}} > \frac{n}{\varepsilon^3}$$

where $c_1 > 0$ is some numerical constant.

Proof: Denote $\alpha = k/n$.
Case 1: $\alpha < 2$. In this case, $1 + \log\left(1 + \frac{k}{n}\right) < 3$, and for $c_1 > 9$,

$$N_k^{c_1 \frac{1+\log\left(1+\frac{2}{\varepsilon}\right)}{1+\log\left(1+\frac{k}{n}\right)}} > N_k^{3+3\log\left(1+\frac{2}{\varepsilon}\right)} > N_k \cdot N_k^{3\log\left(1+\frac{2}{\varepsilon}\right)} > \frac{N_k}{\varepsilon^3} \ge \frac{n}{\varepsilon^3}$$

since for $k \ge 2$ we always have $N_k \ge n \ge 3$.
Case 2: $\alpha \ge 2$. In this case, $1 + \log\left(1 + \frac{k}{n}\right) < 2\log\left(1 + \frac{k}{n-2}\right)$. By (6), $N_k > \left(\frac{n+k-2}{n-2}\right)^{n-2}$. For $c_1 > 6$,

$$N_k^{c_1 \frac{1+\log\left(1+\frac{2}{\varepsilon}\right)}{1+\log\left(1+\frac{k}{n}\right)}} > \left(\left(1 + \frac{k}{n-2}\right)^{n-2}\right)^{\frac{3+3\log\left(1+\frac{2}{\varepsilon}\right)}{\log\left(1+\frac{k}{n-2}\right)}}$$

$$= e^{3(n-2)}\left(1 + \frac{2}{\varepsilon}\right)^{3(n-2)} > \frac{n}{\varepsilon^3}$$

for any $n \ge 3$. $\qquad\square$

Lemma 4.3 *Let $n \ge 3$, and let $k = \alpha n > 0$ be an even number. Then,*

$$\left(\frac{N_k^0}{N_k}\right)^T < \frac{1}{N_k}$$

for $T = c_2\left[1 + \log(1 + \alpha)\right]$, where $c_2 > 0$ is a numerical constant.

Proof: Since $\frac{n+2k-2}{n+k-2} > 1$, it is sufficient to prove that

$$\left(\frac{\binom{n+k/2-2}{k/2}}{\binom{n+k-2}{k}}\right)^T < \frac{1}{\binom{n+k-2}{k}}. \tag{7}$$

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Case 1: $\alpha < \frac{1}{2}$. The left hand side of (7) is equal to:

$$\left(\prod_{i=1}^{k/2} \frac{k/2+i}{k/2+i+n-2}\right)^T < \left(\frac{k}{k+n-2}\right)^{\frac{kT}{2}} \leq \left(\frac{k}{n}\right)^{\frac{kT}{2}}.$$

To obtain (7) it is enough to prove that

$$\left(\frac{k}{n}\right)^{\frac{kT}{2}} < \left(\frac{1}{e}\frac{k}{k+n-2}\right)^k$$

according to (6). Now, because $\alpha = \frac{k}{n} < \frac{1}{2}$, for $T = 8$,

$$\left(\frac{k}{n}\right)^{\frac{kT}{2}} < \left(\frac{k}{n}\right)^k \left(\frac{1}{2}\right)^{\frac{k(T-2)}{2}} < \left(\frac{2}{3e}\frac{k}{n}\right)^k < \left(\frac{1}{e}\frac{k}{k+n-2}\right)^k.$$

Case 2: $\alpha \geq \frac{1}{2}$. Since $\binom{m}{l} = \binom{m}{m-l}$, the left hand side of (7) also equals:

$$\left(\prod_{i=1}^{n-2} \frac{k/2+i}{k+i}\right)^T < \left(\frac{n-2+k/2}{n-2+k}\right)^{(n-2)T} < \left(\frac{5}{6}\right)^{(n-2)T}$$

since $n - 2 < 2k$ and because $\frac{x+k/2}{x+k}$ is an increasing function of x.
Now, for any $T > \frac{1+\log(1+\frac{k}{n-2})}{\log(6/5)}$,

$$\left(\frac{5}{6}\right)^{(n-2)T} < \left(\frac{1}{e}\frac{n-2}{n+k-2}\right)^{(n-2)} < \frac{1}{\binom{n+k-2}{n-2}}.$$

Since for $n \geq 3$, we have $\frac{1+\log(1+\frac{k}{n-2})}{\log(6/5)} < 10\left[1 + \log(1+\alpha)\right]$, the lemma is proved. $\qquad \square$

5 Proof of the Minkowski symmetrization result

We make use of Jackson's theorem for the sphere, due to Newman and Shapiro [NS]:

Theorem 5.1 *Let $n, k > 0$ be integers, and let $f : S^{n-1} \to \mathbb{R}$ be a λ-Lipschitz function on the sphere (i.e. $|f(x) - f(y)| \leq \lambda|x - y|$ for*

any $x, y \in S^{n-1}$). Then there exists a polynomial P_k of degree k in n variables, such that for any $x \in S^{n-1}$,

$$|f(x) - P_k(x)| \leq c_3 \lambda \frac{n}{k}$$

where $c_3 > 0$ is some numerical constant.

Proof of Theorem 1.3: We assume that $M^*(K) = 1$. Begin with $5n$ symmetrizations, according to Theorem 1.2, to obtain a centrally-symmetric body $\bar{K}$. Denote by h its supporting functional. Then h is a norm and hence its Lipschitz constant equals $\sup_{x \in S^{n-1}} h(x)$. By Theorem 1.2,

$$\sup_{x \in S^{n-1}} h(x) < 1 + c \frac{|\log \log n|}{\sqrt{\log n}} < c_4 \tag{8}$$

for some numerical constant $c_4 > 0$. Hence h is a c_4-Lipschitz function, and by Theorem 5.1, there exists a polynomial $P_\varepsilon(x)$ of degree $k = \lceil \frac{n}{\varepsilon} \rceil$ such that,

$$\sup_{x \in S^{n-1}} |P_\varepsilon(x) - h(x)| < c_4 c_3 \varepsilon. \tag{9}$$

Let $P_\varepsilon(x) = \sum_{i=0}^{k} P_i(x)$ be the expansion of P_ε into spherical harmonics. Randomly select T orthonormal bases (i.e. the bases are chosen independently and uniformly in $O(n)$). Apply the corresponding T orthogonal symmetrizations to P_ε and $P_1, .., P_k$, to obtain the random polynomials P'_ε and $P'_1, .., P'_k$. Note that still $P'_\varepsilon = \sum_{i=0}^{k} P'_i$. Successive application of Proposition 3.2 yields that for an even $i > 0$,

$$\mathbb{E}\|P'_i\|_2^2 = \left(\frac{N_i^0}{N_i}\right)^T \|P_i\|_2^2.$$

Combining this with Lemma 4.1 (assume $n \geq 3$),

$$\mathbb{E}\|P'_i\|_\infty^2 \leq N_i \left(\frac{N_i^0}{N_i}\right)^T \|P_i\|_2^2.$$

Assume that $T > (c_1 + 1)c_2 \left[1 + \log\left(1 + \frac{2}{\varepsilon}\right)\right]$. According to Lemma 4.3,

$$\mathbb{E}\|P'_i\|_\infty^2 < N_i \left(\frac{1}{N_i}\right)^{(c_1+1)\frac{1+\log\left(1+\frac{2}{\varepsilon}\right)}{1+\log\left(1+\frac{i}{n}\right)}} \|P_i\|_2^2$$

$$< N_i^{-c_1\frac{1+\log\left(1+\frac{2}{\varepsilon}\right)}{1+\log\left(1+\frac{i}{n}\right)}} \|P_i\|_2^2 < \frac{\varepsilon^3}{n} \|P_i\|_2^2$$

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where the last inequality follows from Lemma 4.2. Denote $I = P_0 = \int_{S^{n-1}} P_\varepsilon(x)d\sigma(x)$. Then,

$$\mathbb{E}\|P'_\varepsilon(x) - I\|_\infty \le \sum_{i=1}^{\lfloor \frac{k}{2} \rfloor} \mathbb{E}\|P'_{2i}(x)\|_\infty \le \sqrt{\frac{k}{2} \sum_{i=1}^{\lfloor \frac{k}{2} \rfloor} \mathbb{E}\|P'_{2i}(x)\|_\infty^2}$$

$$\le \sqrt{\frac{1}{2}\left\lceil \frac{n}{\varepsilon}\right\rceil \sum_{i=1}^{\lfloor \frac{k}{2} \rfloor} \frac{\varepsilon^3}{n}\|P_{2i}(x)\|_2^2} < \varepsilon\|P_\varepsilon\|_2 < \varepsilon\|P_\varepsilon\|_\infty < \varepsilon(c_4 + c_4 c_3 \varepsilon)$$

where the last inequality follows from (8) and (9). Apply the same T orthogonal symmetrizations to $\bar{K}$, and obtain K'. Denote by h' the supporting functional of K'. Then,

$$\sup_{x \in S^{n-1}} |h'(x) - P'_\varepsilon(x)| < c_4 c_3 \varepsilon,$$

and since by (9) we have $1 - c_4 c_3 \varepsilon < I < 1 + c_4 c_3 \varepsilon$, then

$$\mathbb{E} \sup_{x \in S^{n-1}} |h'(x) - 1| < c_4 c_3 \varepsilon + c_4 c_3 \varepsilon + \varepsilon(c_4 + c_4 c_3 \varepsilon) < c'\varepsilon.$$

Clearly,

$$\sup_{x \in S^{n-1}} |h'(x) - 1| < c'\varepsilon \quad \Rightarrow \quad (1 - c'\varepsilon)D \subset K' \subset (1 + c'\varepsilon)D.$$

To summarize, we applied $5n + (c_1 + 1)c_2 \left[1 + \log\left(1 + \frac{2}{\varepsilon}\right)\right] n$ Minkowski symmetrizations to an arbitrary convex body, some of which were chosen randomly. As a result of these symmetrizations, we obtained a body such that the expectation of its distance to a Euclidean ball is no more than $c'\varepsilon$. Therefore, there exists some numerical constant $c > 0$, and $cn \log \frac{1}{\varepsilon}$ symmetrizations that bring the body to be ε-close to a Euclidean ball. $\qquad \square$

Remarks:

1. The case $n = 2$ should be treated separately. In this case, $dim(\mathcal{S}_k) = 2, dim(\mathcal{S}_k^\circ) = 1$ for any k. It is easy to verify that the proof works in this case as well.

2. Theorem 1.3 is optimal in the sense that one cannot obtain an estimate for the number of minimal symmetrizations, of the form $f(n)g(\varepsilon)$ with $f(n) << n$ or $g(\varepsilon) << \log \frac{1}{\varepsilon}$. Indeed, the dependence on n should be at least linear, as it takes a segment $n - 1$ symmetrizations just to become n-dimensional. Regarding the dependence on ε, if we take a segment and apply any $\lfloor c \log \frac{1}{\varepsilon} \rfloor$ symmetrizations, then the segment is transformed into a zonotope which is a sum of no more than $\frac{1}{\varepsilon^c}$ segments. Even in dimension two, this zonotope cannot be ε-close to a Euclidean ball, for a small enough c.

3. Note that Theorem 1.3 is not tight for all possible values of n and ε. For example, Theorem 1.2 is better than Theorem 1.3 when $\varepsilon = c\frac{|\log\log n|}{\sqrt{\log n}}$.

6 Application to Steiner Symmetrization

In this section we prove Theorem 1.5. We make use of a result due to Bokowski and Heil. The following theorem is a special case of Theorem 2 in [BH] (the case $(i, j, k) = (0, d-1, d)$ in the notations of that paper).

Theorem 6.1 *Let $K \subset RD$ be a convex body. Then,*

$$n^2 R^{n-1} M^*(K) \leq \frac{Vol(K)}{Vol(D)} + (n^2 - 1)R^n.$$

An immediate corollary follows:

Corollary 6.2 *Let $\varepsilon > 0$, and let $K \subset (1+\varepsilon)D$ be a convex body in $\mathbb{R}^n$ with $Vol(K) = Vol(D)$. Then,*

$$M^*(K) < 1 + \left(1 - \frac{1}{n^2}\right)\varepsilon.$$

In addition, if $\varepsilon < \frac{1}{n}$ then,

$$M^*(K) < 1 + \left(1 - \frac{1}{2n}\right)\varepsilon.$$

Proof: By Theorem 6.1, since $\frac{Vol(K)}{Vol(D)} = 1$,

$$M^*(K) \leq (1+\varepsilon)\left(1 - \frac{1}{n^2}\right) + \frac{1}{n^2(1+\varepsilon)^{n-1}} < (1+\varepsilon)\left(1 - \frac{1}{n^2}\right) + \frac{1}{n^2}$$

and therefore $M^*(K) < 1 + \left(1 - \frac{1}{n^2}\right)\varepsilon$. Now, assume that $\varepsilon < \frac{1}{n}$. Using the elementary inequality $\frac{1}{(1+\varepsilon)^{n-1}} < 1 - (n-1)\varepsilon + \frac{n(n-1)}{2}\varepsilon^2$, we obtain

$$M^*(K) \leq (1+\varepsilon)\left(1 - \frac{1}{n^2}\right) + \frac{1}{n^2}\left[1 - (n-1)\varepsilon + \frac{n(n-1)}{2}\varepsilon^2\right]$$

$$< 1 + \varepsilon - \frac{\varepsilon}{n} + \frac{\varepsilon^2}{2} < 1 + \varepsilon - \frac{\varepsilon}{2n}.$$

$\square$

Given a convex body $K \subset \mathbb{R}^n$, define $R(K) = \inf\{R > 0; K \subset RD\}$.

Lemma 6.3 *Let $K \subset \mathbb{R}^n$ be a convex body with $Vol(K) = Vol(D)$. Assume that there exists $0 < \varepsilon < C$ such that $R(K) = 1 + \varepsilon$, where $C > 1$. Then there exist $c_5 n(\log \frac{1}{\varepsilon} + \log n)$ Steiner symmetrizations that transform K into $\tilde{K}$ such that*

$$R(\tilde{K}) < 1 + \left(1 - \frac{1}{2n^2}\right)\varepsilon$$

and if $\varepsilon < \frac{1}{n}$,

$$R(\tilde{K}) < 1 + \left(1 - \frac{1}{4n}\right)\varepsilon$$

where $c_5 = c_5(C) > 0$ depends solely on C.

Proof: Let $\tilde{K}$ be the body obtained from K after the $cn \log \frac{4Cn^3}{\varepsilon}$ symmetrizations given by Theorem 1.3. Despite the fact that Theorem 1.3 is concerned with Minkowski symmetrizations, we apply the corresponding Steiner symmetrization (with respect to the same hyperplanes). Since Steiner symmetrizations are contained in Minkowski symmetrizations,

$$R(\tilde{K}) < \left(1 + \frac{\varepsilon}{4Cn^3}\right) M^*(K).$$

Apply corollary 6.2 and the fact that $\varepsilon < C$ to get that

$$R(\tilde{K}) < \left(1 + \frac{\varepsilon}{4Cn^3}\right)\left[1 + \left(1 - \frac{1}{n^2}\right)\varepsilon\right] < 1 + \left(1 - \frac{1}{2n^2}\right)\varepsilon$$

and if $\varepsilon < \frac{1}{n}$,

$$R(\tilde{K}) < \left(1 + \frac{\varepsilon}{4Cn^3}\right)\left[1 + \left(1 - \frac{1}{2n}\right)\varepsilon\right] < 1 + \left(1 - \frac{1}{4n}\right)\varepsilon.$$

$\square$

Proposition 6.4 *Let $n \geq 2$, $0 < \varepsilon < \frac{1}{2}$, and let $K \subset \mathbb{R}^n$ be a convex body with $Vol(K) = Vol(D)$. Then there exist $c_6 \left[n^3 \log^2 n + n^2 \log^2 \frac{1}{\varepsilon}\right]$ Steiner symmetrizations, that transform K into $\tilde{K}$ which satisfies*

$$R(\tilde{K}) < 1 + \varepsilon$$

where $c_6 > 0$ is a numerical constant.

Proof: First, apply $3n$ Steiner symmetrizations to K, according to Theorem 1.4, to obtain an isomorphic Euclidean ball $\bar{K}$. Then,

$$\bar{K} \subset CD.$$

Let us define a sequence of convex bodies: $K_0 = \bar{K}$, and K_i is obtained from K_{i-1} using $c_5 n (\log \frac{1}{R(K_{i-1})-1} + \log n)$ Steiner symmetrizations, as in Lemma 6.3. Then,

$$R(K_i) - 1 < \left(1 - \frac{1}{2n^2}\right)[R(K_{i-1}) - 1] < \left(1 - \frac{1}{2n^2}\right)^i [R(K_0) - 1].$$
(10)

Let T_1 be the minimal integer such that

$$R(K_{T_1}) < 1 + \frac{1}{n}.$$

Since $R(K_0) < C$, then by (10) necessarily $T_1 < cn^2 \log n$. For any $i \le T_1$ we have $R(K_{i-1}) \ge 1 + \frac{1}{n}$ and hence by Lemma 6.3 we used no more than $c'n \log n$ symmetrizations to obtain K_i from K_{i-1}. In total, we used less than $\tilde{c}n^3 \log^2 n$ symmetrizations to obtain K_{T_1}. By Lemma 6.3 for any $i > 0$,

$$R(K_{T_1+i}) - 1 < \left(1 - \frac{1}{4n}\right)[R(K_{T_1+i-1}) - 1] < \left(1 - \frac{1}{4n}\right)^i.$$

Let T_2 be the first integer such that

$$R(K_{T_1+T_2}) < 1 + \varepsilon.$$

Then $T_2 < cn \log \frac{1}{\varepsilon}$. Define $\tilde{K} = K_{T_1+T_2}$. For any $T_1 < i \le T_1 + T_2$ we used no more than $c'n(\log \frac{1}{\varepsilon} + \log n)$ symmetrizations to obtain K_i from K_{i-1}. In total we applied a maximum of $\tilde{c}n^3 \log^2 n + \tilde{c}n^2 \log^2 \frac{1}{\varepsilon}$ Steiner symmetrizations. $\square$

Proposition 6.4 proves the existence of a rather small circumscribing ball for the symmetrized body. In order to symmetrize the body from below, we use the following standard lemma. Its proof is outlined for completeness.

Lemma 6.5 *Let $0 < \varepsilon < 1$, and let $K \subset \mathbb{R}^n$ be a convex body with $M^*(K) \ge 1$. Assume that $K \subset [1 + (c_7\varepsilon)^n]D$. Then*

$$(1 - \varepsilon)D \subset K$$

where $c_7 > 0$ is some numerical constant.

Proof: Assume on the contrary that there exists $x_0 \in S^{n-1}$ with $\|x_0\|_* < 1 - \varepsilon$, where $\|\cdot\|_* = h_K(\cdot)$. Then for $x \in S^{n-1}$ with $|x - x_0| < \frac{\varepsilon}{4}$, we have

$$\|x\|_* \le \|x_0\|_* + \|x - x_0\|_* < 1 - \varepsilon + (1 + (c_7\varepsilon)^n)|x - x_0| < 1 - \frac{\varepsilon}{2}.$$

Denote $A = \{x \in S^{n-1}; |x - x_0| \leq \frac{\varepsilon}{4}\}$. Then,

$$M^*(K) = \int_{S^{n-1}} \|x\|_* d\sigma(x) < (1 - \sigma(A))(1 + (c_7\varepsilon)^n) + \sigma(A)\left(1 - \frac{\varepsilon}{2}\right).$$

The projection of A onto the hyperplane orthogonal to x_0 contains a Euclidean ball of radius larger than $\frac{\varepsilon}{4\sqrt{2}}$. Therefore,

$$\sigma(A) > \frac{Vol(D_{n-1})}{Vol(S^{n-1})}\left(\frac{\varepsilon}{4\sqrt{2}}\right)^{n-1} > \frac{1}{\sqrt{\pi n}}\left(\frac{\varepsilon}{4\sqrt{2}}\right)^{n-1} > \left(\frac{\varepsilon}{30}\right)^{n-1}$$

where D_{n-1} is the $n-1$ dimensional Euclidean unit ball, and Vol is interpreted here as the $n-1$ dimensional volume. Thus,

$$M^*(K) < 1 + (c_7\varepsilon)^n - \sigma(A)\frac{\varepsilon}{2} < 1 + (c_7\varepsilon)^n - \left(\frac{\varepsilon}{30}\right)^n$$

and for $c_7 = \frac{1}{30}$ we obtain a contradiction. $\qquad\square$

Proof of Theorem 1.5: It is sufficient to consider the case $Vol(K) = Vol(D)$. Apply Proposition 6.4 with $\varepsilon' = (c_7\varepsilon)^n$. We use

$$c_6\left[n^3\log^2 n + n^2\log^2\frac{1}{\varepsilon'}\right] < c'n^4\log^2\frac{1}{\varepsilon}$$

Steiner symmetrizations, and obtain a body $\tilde{K}$ such that

$$\tilde{K} \subset (1 + (c_7\varepsilon)^n)D.$$

Since $Vol(K) = Vol(D)$, by Urysohn $M^*(K) \geq 1$. Using Lemma 6.5,

$$(1 - \varepsilon)D \subset \tilde{K} \subset (1 + (c_7\varepsilon)^n)D \subset (1 + \varepsilon)D$$

and the theorem is proved. $\qquad\square$

Remark: Theorem 1.2 is crucial to the proof of Theorem 1.3. Only after obtaining the precise isomorphic statement regarding Minkowski symmetrization, can we prove the sharp almost isometric version. However, in the proof of Theorem 1.5 we may apply weaker estimates than that in Theorem 1.4, and derive the same conclusion. This could be another indication that the powers in Theorem 1.5 are not optimal.

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